**Lösungen zur Übungsserie: Unbestimmtes Integral**

1. (a) $\int \frac{x^3-2x}{4x^2}dx = \frac{1}{4}\int x dx - \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{8}x^2 - \frac{1}{2}\ln|x| + c$

   (b) $\int \frac{(x-2)^2}{\sqrt[3]{x}}dx = \int \frac{x^2-4x+4}{\sqrt[3]{x}}dx = \int \left(x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 4x^{-\frac{1}{3}}\right)dx$

   $$= \frac{1}{\frac{5}{3}+1}x^{\frac{5}{3}+1} - \frac{4}{\frac{2}{3}+1}x^{\frac{2}{3}+1} + \frac{4}{-\frac{1}{3}+1}x^{-\frac{1}{3}+1} + c$$

   $$= \frac{3}{8}x^{\frac{3}{8}} - \frac{12}{5}x^{\frac{5}{3}} + 6x^{\frac{2}{3}} + c = \sqrt[3]{x^2}\left(\frac{3}{8}x^2 - \frac{12}{5}x^3 + 6\right) + c$$

   (c) $\int (\cos x - \frac{n-1}{\cos^2 x})dx = \sin x - (n-1)\tan x + c$

2. Substitution: $\int f(z(x))\ z'(x)\ dx = \int f(z)\ dz$

   (a) $\int \sin^3 x \cdot \cos x\ dx = \int z^3\ \ \cos x\ \frac{dz}{\cos x} = \int z^3 dz = \frac{1}{4}z^4 + c = \frac{1}{4}\sin^4 x + c$

   $$\boxed{\begin{array}{lll} \text{Sustitution:} & z & = \sin x \\ & \frac{dz}{dx} & = \cos x \\ & dx & = \frac{dz}{\cos x} \end{array}}$$

   (b) $\int x\sqrt{x^2+1}\ dx = \int x\sqrt{z}\frac{dx}{2x} = \frac{1}{2}\int \sqrt{z}dz = \frac{1}{2}\ \frac{1}{\frac{1}{2}+1}z^{\frac{1}{2}+1} + c = \frac{1}{3}\sqrt{(x^2+1)^3} + c$

   $$\boxed{\begin{array}{lll} \text{Substitution:} & z & = x^2+1 \\ & \frac{dz}{dx} & = 2x \\ & dx & = \frac{dz}{2x} \end{array}}$$

   (c) $\int \frac{2}{x\ \ln x}\ dx = \int \frac{2}{xz}\ x\ dz = 2\int \frac{1}{z}dz = 2\ln|z| + c = 2\ln|\ln x| + c$

   $$\boxed{\begin{array}{lll} \text{Substitution:} & z & = \ln x \\ & \frac{dz}{dx} & = \frac{1}{x} \\ & dx & = x\ dz \end{array}}$$

   (d) $\int \frac{arc\sin x}{\sqrt{1-x^2}}\ dx = \int \frac{z}{\sqrt{1-x^2}}\ \sqrt{1-x^2}\ dz = \int z dz = \frac{1}{2}z^2 + c = \frac{1}{2}\left(\arcsin x\right)^2 + c$

   $$\boxed{\begin{array}{lll} \text{Substitution:} & z & = \arcsin x \\ & \frac{dz}{dx} & = \frac{1}{\sqrt{1-x^2}} \\ & dx & = \sqrt{1-x^2}dz \end{array}}$$

   (e) $\int \frac{x+1}{x^2+2x+3}\ dx = \int \frac{x+1}{z}\ \frac{dz}{2(x+1)} = \frac{1}{2}\int \frac{1}{z}dz = \frac{1}{2}\ln|z| + c = \frac{1}{2}\ln|x^2+2x+3| + c$

   $$\boxed{\begin{array}{lll} \text{Substitution:} & z & = x^2+2x+3 \\ & \frac{dz}{dx} & = 2x+2 = 2(x+1) \\ & dx & = \frac{dz}{2(x+1)} \end{array}}$$

   (f) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\ dx = \int \frac{1}{\sqrt{1-z^2}}dz = \arcsin z + c = \arcsin e^x + c$

| Substitution: | |
|---|---|
| | $z = e^x$ |
| | $\frac{dz}{dx} = e^x$ |
| | $dx = \frac{dz}{e^x}$ |

3. Partielle Intergration: $\int u \cdot v' \; dx = u \cdot v - \int u' \cdot v \; dx$

   (a) $\int x^2 \cos \; x \; dx = x^2 \sin x - \int 2x \sin x \; dx$

   | partielle Integration: | |
   |---|---|
   | $u = x^2$ | $v' = \cos x$ |
   | $u' = 2x$ | $v = \sin x$ |

   (b) $\int \ln x \; dx = x \ln x - \int \frac{1}{x} x \; dx = x \ln x + \int 1 dx = x \ln x - x + c = x(\ln x - 1) + c$

   | partielle Integration: | |
   |---|---|
   | $u = \ln x$ | $v' = 1$ |
   | $u' = \frac{1}{x}$ | $v = x$ |

   (c) $\int \cos^3 x \; dx = \cos^2 x \sin x + 2 \int \cos x \sin^2 x \; dx$

   | partielle Integration: | |
   |---|---|
   | $u = \cos^2 x$ | $v' = \cos x$ |
   | $u' = -2 \cos x \sin x$ | $v = \sin x$ |

   | Substitution: | |
   |---|---|
   | | $z = \sin x$ |
   | | $\frac{dz}{dx} = \cos x$ |
   | | $dx = \frac{dz}{\cos x}$ |

   $$= \cos^2 \sin x + 2 \int z^2 \; dx = \cos^2 x \sin x + \frac{2}{3} z^3 + c = \cos^2 \sin x + \frac{2}{3} \sin^3 x + c$$
   $$= \sin x (\cos^2 x + \frac{2}{3} \sin^2 x) + c = \sin x (1 - \frac{1}{3} \sin^2 x) + c$$

   (d) $\int \arccos x \; dx = x \arccos x + \int \frac{1}{\sqrt{1-x^2}} \; dx$

   | partielle Integration: | |
   |---|---|
   | $u = \arccos x$ | $v' = 1$ |
   | $u' = -\frac{1}{\sqrt{1-x^2}}$ | $v = x$ |

   | Substitution: | |
   |---|---|
   | | $z = 1 - x^2$ |
   | | $\frac{dz}{dx} = -2x$ |
   | | $dx = -\frac{dz}{2x}$ |

   $$= x \arccos x - \frac{1}{2} \int \frac{1}{\sqrt{z}} \; dz = x \arccos x - \frac{1}{2} \int z^{-\frac{1}{2}} \; dz$$
   $$= x \arccos x - \frac{1}{2} \frac{1}{-\frac{1}{2}+1} \; z^{-\frac{1}{2}+1} + c = x \arccos x - \sqrt{z} + c$$
   $$= x \arccos x - \sqrt{1 - x^2} + c$$

   (e) $\int \frac{x}{e^x} \; dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + c = -\frac{x+1}{e^x} + c$

   | partielle Integration: | |
   |---|---|
   | $u = x$ | $v' = e^{-x}$ |
   | $u' = 1$ | $v = -e^{-x}$ |

   (f) $\int x \cosh x \; dx = x \sinh x - \int \sinh x \; dx = x \sinh x - \cosh x + c$

   | partielle Integration: | |
   |---|---|
   | $u = x$ | $v' = \cosh x$ |
   | $u' = 1$ | $v = \sinh x$ |

4. (a) $\int \frac{2x-1}{x^2-3x+2} \; dx$

Partialbruchzerlegung

Nullstellen des Nenners $x^2 - 3x + 2 = 0 \quad \Rightarrow \quad x_1 = 2,\ x_2 = 1$

$\frac{2x-1}{x^2-3x+2} = \frac{2x-1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)} = \frac{(A+B)x+(-A-2B)}{(x-2)(x-1)}$

Koeffizientenvergleich:

$$\left.\begin{array}{ll} x: & A + \ \ B = 2 \\ 1: & -A - 2B = -1 \end{array}\right\} \quad \Rightarrow \quad B = -1,\ A = 3$$

$\int \frac{2x-1}{x^2-3x+2}\,dx = \int \left(\frac{3}{x-2} - \frac{1}{x-1}\right)\,dx = 3\ln|x-2| - \ln|x-1| + c$

(b) $\int \frac{1}{(x-1)^2(x-2)}\,dx$

Partialbruchzerlegung

$$\begin{aligned} \frac{1}{(x-1)^2(x-2)} &= \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)} = \frac{A(x-1)(x-2)+B(x-2)+C(x-1)^2}{(x-1)^2(x-2)} \\ &= \frac{A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)}{(x-1)^2(x-2)} \end{aligned}$$

Koeffizientenvergleich: $\left.\begin{array}{lrrrr} x^2: & A & & +\ C & = 0 \\ x: & -3A & +\ B & -2C & = 0 \\ 1: & 2A & -2B & +\ C & = 1 \end{array}\right\} \quad \Rightarrow \quad A = -1,\ C = 1,\ B = -1$

$$\begin{aligned} \int \frac{1}{(x-1)^2(x-2)}\,dx &= \int \left(-\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}\right)\,dx = -\ln + \frac{1}{x-1} + \ln|x-2| + c \\ &= \ln\left|\frac{x-2}{x-1}\right| + \frac{1}{x-1} + c \end{aligned}$$

(c) $\int \frac{3x+2}{x^4+3x^3+8x+24}\,dx$

Partialbruchzerlegung

Nullstellen des Nenners: $x^4 + 3x^3 + 8x + 24 = (x+2)(x+3)\underbrace{(x^2-2x+4)}_{\text{komplexe NST}}$

Ansatz:

$$\begin{aligned} \frac{3x+2}{x^4+3x^3+8x+24} &= \frac{3x+2}{x^4+3x^3+8x+24} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{Cx+D}{x^2-2x+4} \\ &= \frac{A(x+3)(x^2-2x+4)+B(x+2)(x^2-2x+4)+(Cx+D)(x+2)(x+3)}{(x+2)(x+3)(x^2-2x+4)} \\ &= \frac{A(x^3+x-2x+12)+B(x^3+8)+C(x^3+5x^+6x)+D(x^2+5x+6)}{(x+2)(x+3)(x^2-2x+4)} \end{aligned}$$

Koeffizientenvergleich:

$$\left.\begin{array}{lrrrrr} x^3: & A & +B & +\ C & & = 0 \\ x^2: & A & & +5C & +\ D & = 0 \\ x: & -2A & & +6C & +5D & = 3 \\ 1: & 12A & +8B & & +6D & = 2 \end{array}\right\} \quad \Rightarrow \quad A = -\frac{19}{57},\ B = \frac{21}{57},\ C = -\frac{2}{57},\ D = \frac{29}{57}$$

$\int \frac{3x+2}{x^4+3x^3+8x+24}\,dx = \frac{1}{57}\left(-\int \frac{19}{x+2}\,dx + 21\int \frac{1}{x+3}\,dx + \int \frac{-2x+29}{x^2-2x+4}\,dx\right)$

$= \frac{1}{57}\left(-19\ln|x+2| + 21\ln|x+3| - \ln|x^2-2x+4| + 9\sqrt{3}\arctan\frac{1}{\sqrt{3}}(x-1)\right) + c$