Lösungen zur Übungsserie: Bestimmtes Integral

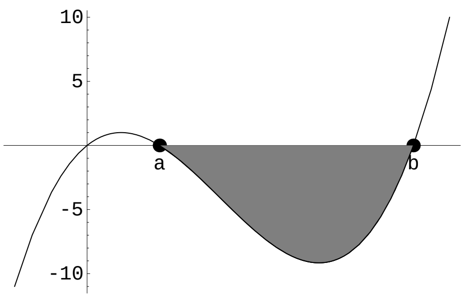
1. $I = \int\limits_a^b x(x-a)(x-b)\ dx, \quad b > a$

$= \int\limits_a^b [x^3 - (a+b)x^2 + abx]\ dx = [\frac{1}{4}x^4 - \frac{a+b}{3}x^3 + \frac{ab}{2}]_a^b = \frac{x^2}{12}[3x^2 - 4(a+b)x + 6ab]_a^b$

$= \frac{b^2}{12}[-b^2 + 2ab] + \frac{a^2}{12}[a^2 - 2ab] = \frac{1}{12}[-b^4 + a^4 + 2ab(b^2 - a^2)] = \frac{1}{12}(b^2 - a^2)[-(b^2 + a^2) + 2ab]$

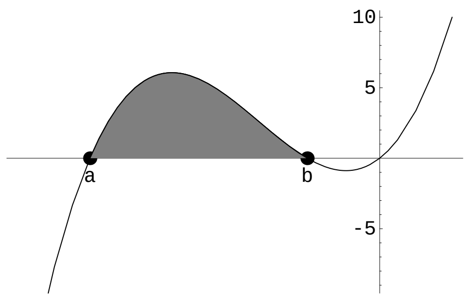
$= -\frac{1}{12}(b^2 - a^2)(b-a)^2$

$0 < a < b$



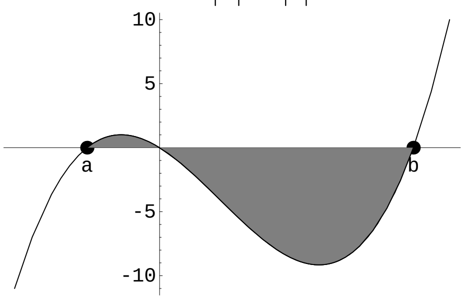
$I < 0$

$a < b < 0$



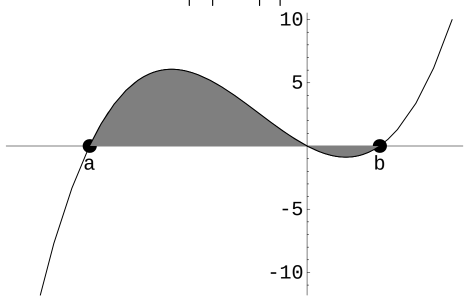
$I > 0$

$a < 0 < b$
und $|a| < |b|$



$I < 0$

$a < 0 < b$
$|a| > |b|$



$I > 0$

2. (a) $\int\limits_0^{\sqrt{2}} \frac{dx}{1+2x^2} = \int \frac{dx}{1+(\sqrt{2}x)^2} = \frac{1}{\sqrt{2}} \int_0^2 \frac{dz}{1+z^2} = \frac{1}{\sqrt{2}} \arctan z|_0^2 = \frac{1}{\sqrt{2}} \arctan 2$
(Substitution: $z = \sqrt{2}x$)

(b) $\int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{\mathrm{e}^{2x}}\ dx$ (2-mal partielle Integration)

$\int \frac{\sin x}{\mathrm{e}^{2x}}\ dx = \int \mathrm{e}^{-2x} \sin x\ dx = -\mathrm{e}^{-2x} \cos x - 2 \int \mathrm{e}^{-2x} \cos x\ dx$

$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\sin x\,dx \qquad \mid +4\int e^{-2x}\sin x\,dx$

$5\int\limits_0^{\frac{\pi}{2}} e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x$

$\int\limits_0^{\frac{\pi}{2}} e^{-2x}\sin x\,dx = -\frac{e^{-2x}}{5}(\cos x + 2\sin x)|_0^{\frac{\pi}{2}} = \frac{1}{5}(1-\frac{2}{e^{\pi}})$

(c) $\int\limits_{x=\frac{1}{2}\sqrt{3}}^{\sqrt{3}} \sqrt{3-x^2}\,dx = \int \sqrt{3}\sqrt{1-(\frac{x}{\sqrt{3}})^2}\,dx = \sqrt{3}\int\limits_{t=\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\sqrt{3}\cos t\,dt = 3\int \cos^2 t\,dt$

(Substitution: $x = \sqrt{3}\sin t$)

$\int \cos^2 t\,dt = \cos t\sin t + \int \sin^2 t\,dt = \cos t\sin t + \int (1-\cos^2 t]\,dt = \cos t\sin t + t - \int \cos^2 t\,dt$

(partielle Integration: $u = \cos x,\ v' = \cos x$)

$2\int \cos^2 t\,dt = \cos t\sin t + t$

$\int\limits_{\frac{1}{2}\sqrt{3}}^{\sqrt{3}} \sqrt{3-x^2}\,dx = \frac{3}{2}[\cos t\sin t + t]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{3}{8}\sqrt{3}$

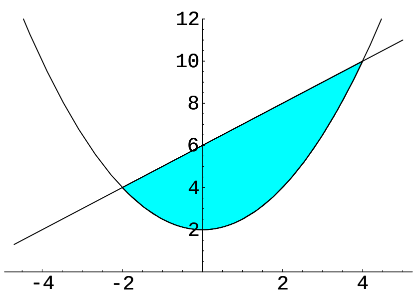
3. Gesucht ist der Inhalt des Flächenstückes, das durch die Kurven $x^2 = 2(y-2)$ und $y = x+6$ eingeschlossen ist?

Schnittpunkte der Kurven:

$y_1(x) = x+6 \quad y_2(x) = 2+\frac{x^2}{2} \quad \Rightarrow \quad x_1 = -2, \quad x_2 = 4$

Fläche:

$F = \int\limits_{-2}^{4} [(x+6) - (2+\frac{x^2}{2})]\,dx = \int (x+4-\frac{1}{2}x^2)\,dx$
$= \frac{1}{2}x^2 + 4x - \frac{1}{6}x^3|_{-2}^{4} = 18\ FE$

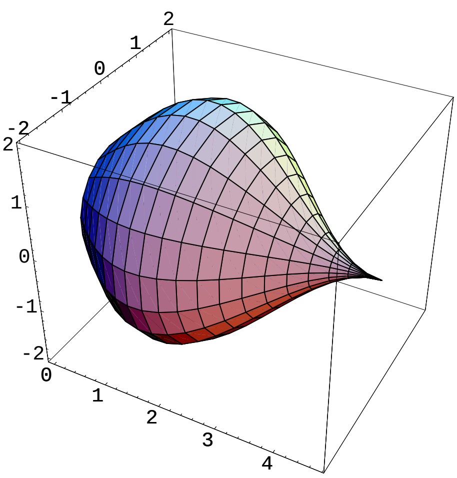


4. Gesucht ist das Volumen des Körpers, der durch Rotation der Kurve $y = 1+\sin\,x$ von $x = 0$ bis $x = \frac{3}{2}\pi$ um die x-Achse entsteht?

Rotationsvolumen:

$= \int\limits_0^{\frac{3}{2}\pi} \pi y^2\,dx = \pi\int (1+\sin x)^2\,dx = \pi\int (1+2\sin x+\sin^2 x)\,dx$

$= \pi[x - 2\cos x + \frac{1}{2}(x - \sin x\cos x)]_0^{\frac{3}{2}\pi} = \pi(\frac{9}{4}\pi + 2)\ FE$



5. Gesucht ist die Bogenlänge der Kurve $y = 2\sqrt{x}$ von $x_0 = 0$ bis $x_1 = 1$.

$x = (\frac{y}{2})^2$ Grenzen: $x = 0 \to y = 0 \quad x = 1 \to y = 2$

Bogenlänge $= s = \int\limits_0^2 \sqrt{1 + (\frac{dx}{dy})^2}\; dy = \int \sqrt{1 + \frac{y^4}{4}}\; dy = \frac{1}{2}\int\limits_0^2 \sqrt{4 + y^2}\; dy$ (Formelsammlung)

$= \frac{1}{4}y\sqrt{4 + y^2} + \ln(y + \sqrt{4 + y^2})|_0^2 = \sqrt{2} + \ln(1 + \sqrt{2})$

6. Gesucht ist der Schwerpunkt der Fläche, die begrenzt ist durch:

$$y = \begin{cases} 2e^{x-2} & \text{für} \quad -1 \le x \le 2 \\ \frac{2}{x-1} & \text{für} \quad 2 \le x \le 3 \end{cases} \quad , \quad x = -1,\ x = 3 \text{ und der x-Achse.}$$

Fläche:

$A = \int\limits_{-1}^{2} 2\mathrm{e}^{x-2}\; dx + \int\limits_{2}^{3} \frac{2}{x-1}\; dx = 2 - \frac{2}{\mathrm{e}^3} + 2\ln 2$

Schwerpunkt (x_s, y_s) :

$x_s = \frac{1}{A}\int xy\; dx = \frac{1}{A}2(1 + \frac{2}{\mathrm{e}^3} + 1 + \ln 2) = 1.699$

$y_s = \frac{1}{A}\int \frac{1}{2}y^2\; dx = \frac{1}{A}(2 - \frac{1}{\mathrm{e}^6}) = 0.608$